GENERALIZED MEANS

BY

FRED B. WRIGHT[1]

1. **Introduction.** The expression "mean value" or "average value" of a real-valued function has several different meanings. In the classical sense it is a number between the bounds of the function, chosen in some specified manner. A mean value for a linear space of functions is usually a linear functional which assigns to each member of the space a mean value in this classical sense. In probability theory, the expected value of a random variable is intuitively an average value. There are other contexts in which the average value of a function is not a number, but is another function. For example, the limit whose existence is asserted by an ergodic theorem may be considered as an average value. Again, the conditional expectation of a random variable, given a subfield of the probability space, is a function which may be regarded as an average value.

In this paper, we shall introduce a very general concept of mean value which will include all of these examples. We have chosen, among several different possibilities, to introduce this idea in a way which seems to be closest to the classical sense and which is perhaps more intuitive. This approach requires some preliminary consideration of topological properties of equivalence relations in topological spaces. This has the advantage of making explicit the connection between the present development and some fundamental ideas in the theory of Boolean algebras. In fact, everything that will be done in this note may be regarded as a "linearization" of Halmos' extension of the Stone Representation Theory to include the notions of existential quantifiers and constants in logic [7; 9]. The principal application we shall make of this linearized version will be in the L^∞-algebra of a finite measure space, where the Boolean prototype may also be employed. From a heuristic point of view, the functional concepts of this paper are to quantifiers and constants as probability theory is to logic.

We have elsewhere [26] related the same concepts from the theory of Boolean algebra to the problems of ergodic theory which center around Poincaré's Recurrence Theorem. In this note, we shall show that certain transformations admit a generalized invariant mean, and that they hence have a nontrivial invariant *finitely additive* measure. This provides an affirmative solution for the invariant measure problem to balance Ornstein's recent negative solution [17].

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2. **Continuity of equivalence relations.** Let X denote any set, and let ϕ denote a binary relation on X. That is, ϕ is a subset of the product space $X\times X$. We shall write $x_1\phi x_2$ to denote $(x_1, x_2)\in\phi$. It is often convenient to regard a relation ϕ as a multiple-valued function from X into itself, or alternatively as a function from X to the family of all subsets of X. We may then denote by $\phi(x)$ the set of all $x'\in X$ such that $x\phi x'$. Similarly, we let $\phi^{-1}E$ denote the set of all $x\in X$ such that $\phi(x)\cap E$ is not empty. Hence $x\in\phi^{-1}E$ if and only if there is an element $x'\in E$ such that $x\phi x'$.

2.1. Definition. Let ϕ be a binary relation in a topological space X. We shall say that ϕ is point-closed (point-compact) if $\phi(x)$ is closed (compact) for each $x\in X$. We say that ϕ is upper semicontinuous if $\phi^{-1}C$ is closed for each closed set $C\subset X$, and we say that ϕ is lower semicontinuous if $\phi^{-1}U$ is open for each open set $U\subset X$. The relation ϕ is said to be continuous if it is both upper and lower semicontinuous.

This terminology is more or less standard; it agrees with Michael [15] with regard to semicontinuity. Michael includes the notion of point-closed relation by restricting the range of his carriers.

In the sequel, we shall be concerned only with equivalence relations in X which display some of the above properties. An equivalence relation is characterized by the properties: (i) $\bigcup_{x\in X}\phi(x)=X$; (ii) either $\phi(x_1)=\phi(x_2)$ or $\phi(x_1)\cap\phi(x_2)=\emptyset$. Each of the sets $\phi(x)$ is called an equivalence class. The family Y of all equivalence classes is called the quotient X/ϕ of X modulo ϕ. The mapping π of X onto Y which assigns to each x the class $\phi(x)$ is called the projection of X or of ϕ. If the space Y is given the strongest topology in which π is continuous, the space Y is the quotient space of X modulo ϕ. The following result is both elementary and well-known.

2.2. Lemma. *Let ϕ be an equivalence relation in X.* (a) *A necessary and sufficient condition that ϕ be point-closed is that $Y=X/\phi$ be a T_1-space.* (b) *A necessary and sufficient condition that ϕ be upper semicontinuous is that the projection π be a closed mapping.* (c) *A necessary and sufficient condition that ϕ be lower semicontinuous is that π be an open mapping.*

These follows from the observation that, for any $E\subset X$, $\phi^{-1}E=\pi^{-1}\pi E$, and from the fact that $F\subset Y$ is open if and only if $\pi^{-1}F$ is open. Another easy fact is the following.

2.3. Lemma. *Let ϕ be an equivalence relation in X.* (a) *If X is a T_1-space and if ϕ is upper semicontinuous, then ϕ is point-closed.* (b) *If X is compact and if ϕ is point-closed, then ϕ is upper semicontinuous.* (c) *If X is a compact Hausdorff space, then ϕ is upper semicontinuous if and only if ϕ is point-closed.*

One other fundamental fact that we shall need is the connection between equivalence relations and algebras of real continuous functions that is given by the general form of the Weierstrass-Stone Theorem. We shall fix some notation that will be used throughout the paper, and state the theorem in a form most useful for our purposes.

Throughout this paper, the word "function" will denote "real valued function." If X is any topological space, we shall let $R(X)$ denote the algebra of all *bounded* continuous functions on X. If $A \subset R(X)$, then $[A]$ denotes the least uniformly closed subalgebra of $R(X)$ which contains A and all constant functions. For each A, let $\rho(A)$ denote the relation in X defined by declaring $x_1\rho(A)x_2$ if and only if $f(x_1)=f(x_2)$ for all $f\in A$. If ϕ is any relation in X, let $B(\phi)$ be the set of all $f\in R(X)$ such that $x_1\phi x_2$ implies $f(x_1)=f(x_2)$.

2.4. Weierstrass-Stone Theorem. *Let X be a compact Hausdorff space.* (a) *If $A\subset R(X)$, then $\rho(A)$ is a point-closed equivalence relation, and* $\rho(A)=\rho([A])$. (b) *If ϕ is a point-closed equivalence relation in X, then* $B(\phi)=[B(\phi)]$. (c) *If A and ϕ are as above, then* $\rho(B(\phi))=\phi$ *and* $B(\rho(A))=[A]$.

This is a restatement of [22, Theorem 5]. It asserts that there is a one-one correspondence between the collection of upper semicontinuous equivalence relations in X and the collection of closed subalgebras of $R(X)$ which contain constant functions. We shall later extend this result to include continuous relations.

If ϕ is a point-closed equivalence relation in a compact Hausdorff space X, and if $g\in R(Y)$, define $f\in R(X)$ by setting $f(x)=g(\pi x)$. This is an isometric isomorphism of $R(Y)$ into $R(X)$, and in fact onto $B(\phi)$. The inverse of this mapping is the Gelfand representation of $B(\phi)$ onto $R(Y)$ [5]. We shall (in every case) denote by $\hat{f}$ the Gelfand transform of f.

3. **Extreme values of functions.** Throughout this section let X be a topological space, and let ϕ be an equivalence relation in X.

3.1. Definition. For each $f\in R(X)$, define two functions Mf and mf on X by setting

$$Mf(x) = \operatorname{lub}\{f(x') \colon x' \in \phi(x)\},$$
$$mf(x) = \operatorname{glb}\{f(x') \colon x' \in \phi(x)\}.$$

3.2. Lemma. *For each $f\in R(X)$ and each real α, the following are true.*

(a) $$\phi^{-1}\{x \in X \colon f(x) \geqq \alpha\} \subset \{x \in X \colon Mf(x) \geqq \alpha\}.$$

(b) $$\phi^{-1}\{x \in X \colon f(x) > \alpha\} = \{x \in X \colon Mf(x) > \alpha\}.$$

(c) $$\phi^{-1}\{x \in X \colon f(x) \leqq \alpha\} \subset \{x \in X \colon mf(x) \leqq \alpha\}.$$

(d) $$\phi^{-1}\{x \in X \colon f(x) < \alpha\} = \{x \in X \colon mf(x) < \alpha\}.$$

(e) *If ϕ is point-compact, then equality holds in* (a) *and* (c).

Proof. Suppose that $x_0\in\phi^{-1}\{x\in X\colon f(x)\geqq\alpha\}$; then $x_0\in\phi(x_1)$ for some x_1 such that $f(x_1)\geqq\alpha$. Since ϕ is an equivalence relation, $x_1\in\phi(x_0)$, and hence $Mf(x_0)\geqq f(x_0)\geqq\alpha$. This proves (a). Suppose that ϕ is point-compact. Then f assumes its maximum on the compact set $\phi(x_0)$, say at the point x_1. If

$Mf(x_0) \geqq \alpha$, then $Mf(x_0) = f(x_1) \geqq \alpha$. Since $x_1 \in \phi(x_0)$ and since ϕ is an equivalence relation, then $x_0 \in \phi(x_1)$. This proves the part of (e) that relates to (a). The rest of the proofs are so nearly identical that they may be omitted. (We may remark that the identity $mf = -M(-f)$ enables us to cut some proofs by half.)

3.3. Corollary. *Let ϕ be point-compact.* (a) *If ϕ is upper semicontinuous, then Mf is an upper semicontinuous function and mf is a lower semicontinuous function.* (b) *If ϕ is lower semicontinuous, then Mf is lower semicontinuous and mf is upper semicontinuous.* (c) *If ϕ is continuous, then both Mf and mf are continuous.*

We shall make important use of part (c) of this corollary. The remainder of this section is devoted to the converse of this proposition.

3.4. Theorem. *Let ϕ be an equivalence relation on a completely regular space X. If Mf is continuous for each $f \in R(X)$, then ϕ is lower semicontinuous.*

Proof. We prove, by contradiction, that the projection is open. Suppose that it is not; then there exists an open set $U \subset X$ such that πU is not open in Y, and hence $Y - \pi U$ is not closed. Therefore there is an $x_0 \in U$ such that, if $y_0 = \pi x_0$, any open neighborhood N of y_0 intersects $Y - \pi U$. Because of the complete regularity of X, there is a function $f \in R(X)$ with $0 \leqq f \leqq 1$, $f(x_0) = 1$, $f(x) = 0$ for $x \notin U$. Then $\{x \in X: f(x) > 0\} \subset U$, and therefore $\pi\{x: f(x) > 0\} \subset \pi U$. By 3.2(b), $\pi^{-1}\pi\{x: f(x) > 0\} = \{x: Mf(x) > 0\}$, and hence $\pi\{x: f(x) > 0\} = \pi\{x: Mf(x) > 0\}$. Moreover $1 = f(x_0) \leqq \text{lub}\{f(x): x \in \phi(x_0)\} = Mf(x_0) = 1$. Now $\{x: Mf(x) > 0\} = \pi^{-1}\pi\{x: Mf(x) > 0\}$, and this is open, since Mf is continuous. By the definition of the topology in Y, $\pi\{x: Mf(x) > 0\}$ is open. We have seen that this set contains y_0, and does not intersect $Y - \pi U$. This contradiction shows that π is an open mapping.

3.5. Theorem. *Let X be a normal Hausdorff space, and let ϕ be a point-closed equivalence relation in X. If, for each $f \in R(X)$, Mf is continuous, then ϕ is upper semicontinuous.*

Proof. Let C be a closed set in X, and suppose $x_0 \notin \phi^{-1}C$. Then $\phi(x_0)$ and $\phi^{-1}C$ are disjoint, for if $x_0 \phi x_1$ and $x_1 \phi x_2$, $x_2 \in C$, then $x_0 \phi x_2$, and hence $x_0 \in \phi^{-1}C$. In particular, $\phi(x_0)$ and C are disjoint. Since ϕ is point-closed, $\phi(x_0)$ is closed. Therefore there is an $f \in R(X)$, $0 \leqq f \leqq 1$, $f(x) = 0$ for $x \in \phi(x_0)$, $f(x) = 1$ for $x \in C$. Then $Mf(x) = 0$ for $x \in \phi(x_0)$ and $Mf(x) = 1$ for $x \in \phi^{-1}C$. Since Mf is continuous, there is an open set containing $\phi(x_0)$ and not intersecting $\phi^{-1}C$. In particular, x_0 is not in the closure of $\phi^{-1}C$. Since x_0 was any point not in $\phi^{-1}C$, then $\phi^{-1}C$ is closed.

3.6. Theorem. *Let X be a normal Hausdorff space and let ϕ be an equivalence relation in X.* (a) *If ϕ is point-closed and if Mf is continuous for every*

$f \in R(X)$, then ϕ is continuous. (b) *If ϕ is point-compact and continuous, then Mf is continuous for each $f \in R(X)$.*

3.7. THEOREM. *If X is a compact Hausdorff space and if ϕ is an equivalence relation in X, then ϕ is continuous if and only if Mf is continuous for each $f \in R(X)$.*

These follow at once from the previous two results, and from 2.3 and 3.3.

4. **Relatively complete subalgebras.** If ϕ is a continuous equivalence relation in a compact Hausdorff space X, then Mf and mf both belong to $B(\phi)$. Thus M and m may be regarded as mappings of $R(X)$ into itself. In this section, we investigate some properties of these mappings, and extend the Weierstrass-Stone Theorem to identify the algebras which correspond to continuous equivalence relations.

4.1. DEFINITION. A closed subalgebra B of $R(X)$ will be called relatively complete if and only if B contains all constant functions, and for every $f \in R(X)$, the set $\{g \in B : f \leqq g\}$ contains a least element.

4.2. THEOREM. *Let X be a compact Hausdorff space and let ϕ be a point-closed equivalence relation in X. Then ϕ is continuous if and only if $B(\phi)$ is a relatively complete subalgebra of $R(X)$.*

Proof. If ϕ is continuous, then $Mf \in B(\phi)$, and hence Mf has the required properties, so that $B(\phi)$ is relatively complete.

Conversely, suppose that $B(\phi)$ is relatively complete; we must show that ϕ is lower semicontinuous. Let $f \in R(X)$, and let g be the least element of $B(\phi)$ such that $f \leqq g$. Then $Mf(x) \leqq g(x)$ for all x, and Mf is an upper semicontinuous function. Hence the set $U = \{x \in X : Mf(x) < g(x)\}$ is an open set. It is also clear that $U = \phi^{-1}U = \pi^{-1}\pi U$. By the definition of the topology in the quotient space Y, the set πU is open in Y. Define a function h on Y by setting $h(y) = Mf(\pi^{-1}y)$; this is well-defined, and h is upper semicontinuous on Y, by 3.2(b). Suppose $x_0 \in U$, and let $y_0 = \pi x_0$. Then $h(y_0) < \hat{g}(y_0)$ and $h(y) = \hat{g}(y)$ for all $y \in Y - \pi U$. Define a new function k on Y by setting $k(y) = \hat{g}(y)$ for $y \neq y_0$, and by setting $h(y_0) < k(y_0) < \hat{g}(y_0)$. If $k(y_0) = \alpha$, then, for any real β, $\{y : k(y) > \beta\} = \{y : \hat{g}(y) > \beta\}$ if $\beta < \alpha$, and $\{y : k(y) > \beta\} = \{y : \hat{g}(y) > \beta\} - \{y_0\}$ if $\beta \geqq \alpha$. Hence k is a lower semicontinuous function on Y, with $h(y) \leqq k(y)$ for all y. Therefore there is a continuous function $\hat{g}_1$ on Y such that $h(y) \leqq \hat{g}_1(y) \leqq k(y)$ for all y. Define g_1 on X by $g_1(x) = \hat{g}_1(\pi x)$. Then $g_1 \in B(\phi)$, $g_1(x_0) = \hat{g}_1(y_0) < \hat{g}(y_0) = g(x_0)$, and $f(x) \leqq Mf(x) \leqq g_1(x)$, for all x. This contradicts the choice of g. Therefore the supposition that there is an element $x_0 \in U$ is untenable. Hence U is empty, and $Mf(x) = g(x)$ for all x. Therefore Mf is continuous, and hence ϕ is continuous.

The following is a list of properties the transformation M has when ϕ is continuous. These are immediate consequences of the definition, and proofs will be omitted.

4.3. THEOREM. *Let ϕ be a continuous equivalence relation in a compact Hausdorff space X, let f, $g \in R(X)$, and let $\alpha \geqq 0$ be a real number.* (a) $f \leqq Mf \equiv MMf$. (b) $f = Mf$ *if and only if* $f \in B(\phi)$. (c) $M(f \vee g) = Mf \vee Mg$. (d) $M(f \wedge Mg) = Mf \wedge Mg$. (e) $M(\alpha f) = \alpha Mf$.

4.4. THEOREM. *If ϕ is a continuous equivalence relation in a compact Hausdorff space X and if f, $g \in R(X)$, then $\|f-g\| < \epsilon$ implies $\|Mf - Mg\| < \epsilon$.*

Proof. Fix $x_0 \in X$; since ϕ is point-compact, there is an x_1 in $\phi(x_0)$ such that $f(x_1) = Mf(x_0)$. If $\|f-g\| < \epsilon$, then $f(x_1) - \epsilon < g(x_1) < f(x_1) + \epsilon$. Hence $Mf(x_0) - \epsilon < g(x_1) \leqq Mg(x_1) = Mg(x_0)$. Interchanging the roles of f and g, we obtain $Mg(x_0) - \epsilon < Mf(x_0)$. Combining these two, we have $Mf(x_0) - \epsilon < Mg(x_0) < Mf(x_0) + \epsilon$. Since x_0 was arbitrary, we have $\|Mf - Mg\| < \epsilon$.

5. **The existence of continuous relations.** The existence of upper semicontinuous equivalence relations in a compact Hausdorff space is a trivial problem, and in any event the Weierstrass-Stone Theorem assures a large number. Lower semicontinuity is a more difficult requirement to satisfy, in general. There are always two trivial continuous equivalence relations, which are the extreme cases. The relation ϕ in which $x_1 \phi x_2$ for each x_1, $x_2 \in X$ is called the *simple* equivalence relation. The relation ϕ in which $x_1 \phi x_2$ if and only if $x_1 = x_2$ is called the *discrete* equivalence relation. (This terminology is from Halmos [7].) These relations are continuous in any topological space.

5.1. LEMMA. *In a compact Hausdorff space X, if ϕ is a continuous equivalence relation, then M is linear if and only if ϕ is discrete.*

Proof. If $x_1 \phi x_2$, $x_1 \neq x_2$, let $f \in R(X)$ be such that $f(x_1) = 1$, $f(x_2) = -1$, $-1 \leqq f \leqq 1$. Then $M(f + (1-f)) = M1 = 1$, while $Mf(x_1) + M(1-f)(x_1) = 2$. Hence M is not linear. The converse is trivial.

Certain spaces admit obviously continuous equivalence relations. If X is the cartesian product of compact Hausdorff spaces, define $x_1 \phi x_2$ if and only if all of the coordinates of x_1 and x_2 which come from a (fixed) finite family of the spaces are the same. If G is a locally compact Hausdorff group and if H is a closed subgroup of G, the equivalence relation $x_1 \equiv x_2 \pmod{H}$ is continuous. An illuminating example is given by choosing X to be the interval $[-1, 1]$ of reals, and by defining $x_1 \phi x_2$ to mean $|x_1| = |x_2|$. Here $Y = [0, 1]$, $\pi x = |x|$, and $B(\phi)$ is the algebra of all even functions on X.

The most important case we shall consider in the sequel is that of a totally disconnected, compact Hausdorff space; that is, a so-called Boolean space. For this case, we have the existing Boolean theory, due to Halmos [7], which is the model for the present treatment, and which we shall subsequently employ. A few comments on the connections between the two theories are in order.

Let $\mathfrak{A}$ be any Boolean algebra, and let 2 denote the simple, two-element Boolean algebra: $2 = \{0, 1\}$. We regard 2 as a discrete topological space. The

Stone Representation Theorem asserts that for each $\mathfrak{A}$, there is a Boolean space X such that $\mathfrak{A}$ is isomorphic with the algebra $2(X)$ of all continuous functions from X to 2 [21]. An (existential) quantifier on a Boolean algebra $\mathfrak{A}$ is a mapping $\exists$ of $\mathfrak{A}$ into itself with the properties: (i) $\exists 0=0$; (ii) $p\leqq \exists p$ for each $p\in\mathfrak{A}$; (iii) $\exists(p\wedge\exists q)=\exists p\wedge\exists q$, for each p, $q\in\mathfrak{A}$ [7]. A Boolean relation ϕ in a Boolean space X is a point-closed (binary) relation such that $\phi^{-1}P$ is open-closed for every open-closed $P\subset X$ [7; 25]. It follows from [7, Lemma 16] that ϕ is a Boolean relation in X if and only if ϕ is continuous in the sense of this note. (The algebraic reasons for defining a Boolean relation in the above manner, rather than as a continuous relation, can be seen in [25].)

There is a one-one correspondence between the quantifiers in $\mathfrak{A}$ and the Boolean equivalence relations in X [7, Theorem 10]. Representing $\mathfrak{A}$ as $2(X)$, this correspondence can be described by the relation $\exists p(x) =\text{lub}\{p(x_1): x_1\in\phi(x)\}$. Furthermore, call a Boolean subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ relatively complete if, for each $p\in\mathfrak{A}$, the set $\{q\in\mathfrak{B}: p\leqq q\}$ has a least element. There is a one-one correspondence between the quantifiers in $\mathfrak{A}$ and the relatively complete subalgebras of $\mathfrak{A}$ [7, Theorems 4.5]. If $\mathfrak{A}$ is a complete Boolean algebra, then a Boolean subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is a relatively complete subalgebra if and only if $\mathfrak{B}$ is a complete Boolean algebra (in its own right, not necessarily as a subalgebra of $\mathfrak{A}$). Recall that $\mathfrak{A}$ is complete if and only if X is extremally disconnected, in the sense that the closure of every open set is open. In any case, if $\mathfrak{A}$ is a Boolean algebra, if $\mathfrak{B}$ is a relatively complete subalgebra of $\mathfrak{A}$, and if Y is the Boolean space of $\mathfrak{B}$, then Y is the quotient space of X modulo the Boolean relation ϕ associated with the quantifier determined by $\mathfrak{B}$.

If X is any compact Hausdorff space, Boolean or not, let $E(X)$ denote the set of idempotents in $R(X)$. Then $E(X)$ as a sublattice of $R(X)$ is a Boolean algebra. If ϕ is a continuous equivalence relation in X, and if $e\in E(X)$, then it is clear that $Me\in E(X)$, and 4.3 shows that M restricted to $E(X)$ is a quantifier on $E(X)$. If $E_\phi=E(X)\cap B(\phi)$, then E_ϕ as a sublattice of $B(\phi)$ is the relatively complete Boolean subalgebra of $E(X)$ associated with this quantifier. Now let $D(X)=[E(X)]$ be the smallest uniformly closed subalgebra of $R(X)$ containing $E(X)$. If $\rho=\rho(D(X))$ is the point-closed equivalence relation determined by $D(X)$, and if X^* is the quotient space of X modulo ρ, then X^* is totally disconnected; X^* may in fact be identified with the Boolean space of the Boolean algebra $E(X)$. If $[E_\phi]$ is represented as a subalgebra of $R(X^*)$, then $[E_\phi]$ induces a continuous equivalence relation in X^*.

A necessary and sufficient condition that a compact Hausdorff space X be a Boolean space is that $[E(X)]=R(X)$. In this case, then X is the Boolean space for the Boolean algebra $E(X)$. We shall be interested in the case where X is extremally disconnected, and hence in the case where $E(X)$ is complete.

Under this hypothesis, the relatively complete subalgebras of $R(X)$ can be given an alternative characterization.

5.2. DEFINITION. If X is any topological space, a subset L of $R(X)$ is called a conditionally complete lattice if any subset of L which has an upper and a lower bound in L has both a least upper bound and a greatest lower bound in L.

5.3. THEOREM. *Let X be a Boolean space.* (a) *A necessary and sufficient condition that $R(X)$ be a conditionally complete lattice is that X be extremally disconnected.* (b) *If X is extremally disconnected, and if B is a uniformly closed subalgebra of $R(X)$ containing constant functions, then B is a relatively complete subalgebra of $R(X)$ if and only if B is a conditionally complete lattice.*

Proof. Statement (a) is a result of Stone [23]. To prove (b), suppose first that B is relatively complete. Then $\phi=\rho(B)$ is continuous, and therefore the quotient space Y is a Boolean space. In particular, B is generated by $B\cap E(X)$. As a Boolean algebra, $B\cap E(X)$ is the range of the quantifier induced in $E(X)$ by the relation ϕ, and therefore is relatively complete in $E(X)$. Since $E(X)$ is complete, so is $B\cap E(X)$. This means that Y is extremally disconnected, and by (a), $R(Y)$ is conditionally complete. Since B is isomorphic with $R(Y)$, B is conditionally complete. That conditional completeness implies relative completeness is trivial, even if X is not extremally disconnected.

6. **Generalized means.** In this section, we return to the case of a general compact Hausdorff space X, and let ϕ denote a point-closed equivalence relation in X.

6.1. DEFINITION. A mean for $R(X)$ with respect to ϕ is a linear transformation μ of $R(X)$ into itself with the properties: (i) for each $f\in R(X)$, $\mu f\in B(\phi)$; (ii) for each $f\in R(X)$, we have $mf(x)\leqq\mu f(x)\leqq Mf(x)$ for all $x\in X$.

We note that if ϕ is the simple equivalence relation, this approach reduces to the one of Day [4].

Unless clarity demands otherwise, we shall usually say "mean" rather than "mean for $R(X)$ with respect to ϕ."

6.2. LEMMA. *Let μ be a mean, and let $f, g\in R(X)$.* (a) *$f\in B(\phi)$ if and only if $\mu f=f$.* (b) *If $f\leqq g$, then $\mu f\leqq\mu g$.* (c) *$M\mu f=\mu f$.* (d) *$\mu^2=\mu$.* (e) *$\|\mu\|=1$.*

These are trivial, except possibly for (b). If $f\leqq g$, then $0\leqq g-f$, so $0\leqq m(g-f)\leqq\mu(g-f)=\mu g-\mu f$, and hence $\mu f\leqq\mu g$.

6.3. THEOREM. *Let ϕ be continuous, and let μ be a linear transformation of $R(X)$ onto $B(\phi)$. Then the following are equivalent statements:* (a) *μ is a mean;* (b) *μ is order-preserving and $\mu M=M$;* (c) *μ is order-preserving and $\mu^2=\mu$.*

Proof. If ϕ is continuous, then $Mf, mf\in B(\phi)$ for all $f\in R(X)$. Then (a) implies (b) and (c), by 6.2. Suppose (b) is true; then, since $mf\leqq f\leqq Mf$, we have $\mu mf\leqq\mu f\leqq\mu Mf$, and hence $mf\leqq\mu f\leqq Mf$. Finally, (c) implies (b), for if $\mu^2=\mu$, then $\mu f = f$ for all $f \in B(\phi)$, and hence $\mu Mf = Mf$ for all $f \in R(X)$.

Let $R(X)^*$ be the conjugate space of $R(X)$ as a Banach space. If $\sigma \in R(X)^*$ and if $f \in R(X)$, we denote by $\langle f, \sigma \rangle$ the value of the functional σ on f. We shall also let σ denote the regular, finite, signed, Borel measure in X which is associated with the linear functional by the Riesz Representation Theorem [13; 18]. Then $\langle f, \sigma \rangle = \int_X f(x) d\sigma(x)$. A necessary and sufficient condition that σ be a non-negative measure is that $\|\sigma\| = \sigma(X) = \langle 1, \sigma \rangle$. We shall denote by $P(X)$ the set of all non-negative measures σ in $R(X)^*$ such that $\sigma(X) = 1$, and shall call the elements of $P(X)$ the regular probabilities in X. If $R(X)^*$ is given its weak* topology, then $P(X)$ is a closed set in this topology and is therefore compact, by the Alaoglu theorem [1]. We shall always consider $P(X)$ as a topological space in this topology. The set $P(X)$ is also convex, and hence is the smallest weak*-closed convex set containing the extreme points of $P(X)$, by the Krein-Milman Theorem [14]. The extreme points of $P(X)$ are the 2-valued elements of $P(X)$: σ is called 2-valued if $\sigma(E) = 0$ or $\sigma(E) = 1$ for every Borel set $E \subset X$. An element $\sigma \in P(X)$ is said to be supported by a Borel set $E \subset X$ if $\sigma(F) = \sigma(E \cap F)$ for every Borel set $F \subset X$. An element of $P(X)$ is 2-valued if and only if it has some singleton set $\{x\}$ as support. This one-one correspondence between extreme points of $P(X)$ and points of X is a homeomorphism. Accordingly, we shall regard X as a subset of $P(X)$. If $x \in X$ is so considered, then $\langle f, x \rangle = f(x)$.

6.4. Definition. Let τ be a continuous mapping of a compact Hausdorff space X onto a compact Hausdorff space Y. By a probability section of τ is meant a continuous mapping σ: $Y \to P(X)$ which has the property that, for each $y \in Y$, σ_y is supported by $\tau^{-1}y$.

This is a generalization of the familiar concept of a cross-section of τ. A cross-section of τ is a continuous mapping γ: $Y \to X$ such that $\tau\gamma y = y$ for each $y \in Y$.

6.5. Theorem. *Let ϕ be a point-closed equivalence relation in a compact Hausdorff space X, and let π be the projection of X onto the quotient space Y of X modulo ϕ. If μ is a mean for $R(X)$ with respect to ϕ, then there exists a probability-section σ of π such that $\mu f(x) = \int_X f(\xi) d\sigma_{\pi x}(\xi) = \langle f, \sigma_{\pi x} \rangle$, for each $f \in R(X)$ and each $x \in X$. Conversely, if σ is a probability-section of π, this equation defines a mean for $R(X)$ with respect to ϕ.*

Proof. Suppose first that σ is a probability-section of π. For each $f \in R(X)$ we define a function νf on Y by setting $\nu f(y) = \int_X f(\xi) d\sigma_y(\xi) = \langle f, \sigma_y \rangle$. The continuity of the mapping σ implies that νf is continuous on Y, and it is obvious that ν is a linear transformation of $R(X)$ into $R(Y)$. For each $x \in \pi^{-1}y$ we have $mf(x) \leq f(x) \leq Mf(x)$, so that

$$\int_{\pi^{-1}y} mf(\xi) d\sigma_y(\xi) \leq \int_{\pi^{-1}y} f(\xi) d\sigma_y(\xi) \leq \int_{\pi^{-1}y} Mf(\xi) d\sigma_y(\xi).$$

Since σ_y is supported by $\pi^{-1}y$ and since $\sigma_y(X)=1$, and further since Mf and mf are constant on $\pi^{-1}y$, we have $mf(x)\leqq \nu f(y)\leqq Mf(x)$ for all $x\in\pi^{-1}y$. Define $\mu f(x)=\nu f(\pi x)$; μ is then a mean.

Conversely, suppose μ is a mean. Then μ maps $R(X)$ onto $B(\phi)$. Let ν be the linear transformation of $R(X)$ onto $R(Y)$ obtained by composing μ with the Gelfand representation of $B(\phi)$ onto $R(Y)$. Then ν has an adjoint transformation ν^*: $R(Y)^*\rightarrow R(X)^*$. Let σ denote the restriction of ν^* to the set Y of extreme points of $P(Y)$. Then σ is continuous on Y to $R(X)^*$ in the weak* topology. Since $\|\nu\|=1$, then $\|\nu^*\|=1$, and hence $\|\sigma_y\|\leqq 1$. But $\mu 1=1$, so $\nu 1=1$, and hence $\sigma_y(X)=1$. Thus $\sigma_y\in P(X)$. From the definition of adjoints, we have $\langle f, \sigma_{\pi x}\rangle=\langle \nu f, \pi x\rangle=\nu f(\pi x)=\mu f(x)$, the latter following from the definition of ν. It remains to show that σ_y is supported by $\pi^{-1}y$. (This much of the theorem is standard [3] representation theory for linear transformations with range $R(Y)$; clearly we have not yet used the peculiar properties of a generalized mean.)

Let $f\in R(X)$ be a function which vanishes on $\pi^{-1}y$. Then $mf(x)=Mf(x)=0$ for all $x\in\pi^{-1}y$, and hence $\mu f(x)=0$ if $x\in\pi^{-1}y$. Then $\nu f(y)=0$, so that $\langle f, \sigma_y\rangle=\int_X f(\xi)d\sigma_y(\xi)=0$. Now if C is any closed set disjoint from $\pi^{-1}y$, there is an $f\in R(X)$ which vanishes on $\pi^{-1}y$ and which is equal to 1 on C. Hence $0\leqq\sigma_y(C)\leqq\int_X f(\xi)d\sigma_y(\xi)=0$, so that $\sigma_y(C)=0$ for any closed set disjoint from $\pi^{-1}y$. By the regularity of σ_y, we then have $\sigma_y(E)=0$ for any Borel set E which is disjoint from $\pi^{-1}y$. Hence σ_y is supported by $\pi^{-1}y$, and the proof is complete.

If ϕ is the simple equivalence relation, then 6.5 reduces to the Riesz Representation Theorem. If ϕ is the discrete equivalence relation then $B(\phi)=R(X)$, $Y=X$, and π and μ are the identity mappings. The discrete equivalence relation arises from any separating family of functions in $R(X)$. One may say, then, that in some sense the Riesz Representation Theorem and the Weierstrass-Stone Theorem on separating families are the extreme cases of Theorem 6.5.

6.6. Definition. A homomorphic mean for $R(X)$ with respect to ϕ is a mean which is also a homomorphism of $R(X)$ onto $B(\phi)$.

6.7. Theorem. *A necessary and sufficient condition that a mean be homomorphic is that its associated probability section be a cross-section.*

Proof. A mean μ is a homomorphism if and only if $\sigma_{\pi x}$ is a homomorphism of $R(X)$ into the real numbers, for each $x\in X$. It is well known that the set of such homomorphisms is the set X regarded as a subset of $R(X)^*$. Hence μ is a homomorphism if and only if σ is a cross-section.

More generally, if η is a homomorphism of $R(X)$ onto $R(Y)$, with $\eta 1=1$, where X and Y are compact Hausdorff spaces, then there is a continuous mapping γ: $Y\rightarrow X$ such that $\eta f(y)=f(\gamma y)$. If Y is the quotient space of X modulo a point-closed equivalence relation ϕ, then γ is a cross-section of the projection π if and only if $\pi\gamma\pi x=\pi x$ for all $x\in X$. This holds if and only if $(\gamma\pi x)\phi x$, which in turn holds if and only if $f\in B(\phi)$ implies $f(\gamma\pi x)=f(x)$. If we regard η as a homomorphism of $R(X)$ onto $B(\phi)$, this last condition is equivalent to requiring that $\eta f=f$ for each $f\in B(\phi)$, which holds if and only if $\eta^2 f=\eta f$ for each $f\in R(X)$. We state the conclusion as a theorem.

6.8. THEOREM. *Let η be a homomorphism of $R(X)$ into itself. Then η is a homomorphic mean if and only if $\eta 1=1$ and $\eta^2=\eta$.*

In calculus, a "mean value theorem" is a result stating that a mean can be realized as a functional value. Theorems 6.5 and 6.7 may be regarded as generalized mean value theorems.

7. **The existence problem.** It is not a trivial matter to prove that a point-closed equivalence relation in a compact Hausdorff X has any means. For the case of the simple equivalence relation, each point in X furnishes a homomorphic mean. It is an elementary but profound fact that for any given f there is a point x in X such that $f(x)=Mf$. The analogous question for the general point-closed equivalence relation ϕ is this: given an $f\in R(X)$, is there a homomorphic mean μ such that $\mu f=Mf$? More generally, is there any mean μ such that $\mu f=Mf$? In the section following, we will consider an important special case in which some affirmative answers can be given.

There exist some relations, even continuous ones, which admit no cross-sections. (See [2] for an example.) Even if cross-sections exist, the answer to the first question may be negative. Consider the function $f(x, y)=(2x-1)(2y-1)$ on the product of two unit intervals, and let the relation be given by the projection on one of the factors. Consideration of this function, together with the representation Theorem 6.5 shows that the answer to the second question is also negative, in general.

The set of means is easily seen to be a convex set in the Banach algebra of bounded linear transformation of $R(X)$ into $B(\phi)$. If this algebra is given the weak operator topology, the set of means is compact. The Krein-Milman Theorem then applies, and the existence of means is reduced to the existence of extreme means. It is easy to see that any homomorphic mean is an extreme point of the set of means. Question: Is every extreme point of the set of means a homomorphism? An affirmative answer, even in special cases, would be of use in proving the existence of cross-sections, since it is presumably easier to construct linear transformations than it is to construct homomorphisms.

8. **Conditional means.** If Ω is any set and if $\mathfrak{A}$ is any family of subsets of Ω containing Ω and closed under countable unions and under complementation, we shall call $\mathfrak{A}$ a Borel field in Ω. A Borel field $\mathfrak{B}$ in Ω which is contained in $\mathfrak{A}$ will be called a Borel subfield of $\mathfrak{A}$. A probability space $(\Omega, \mathfrak{A}, P)$ is a set Ω, a Borel field $\mathfrak{A}$ in Ω, and a countably additive measure P on $\mathfrak{A}$ such that $P(\Omega)=1$. We denote by $\mathfrak{J}$ the σ-ideal of all sets $A\in\mathfrak{A}$ such that $P(A)=0$. The quotient algebra $\mathfrak{A}=\mathfrak{A}/\mathfrak{J}$ will be called the measure algebra of the probability space. We denote by $L^\infty(\mathfrak{A})$ the Banach algebra of all essentially bounded

functions on Ω which are measurable $(\mathfrak{A})$, with norm denoted by $\|F\|_\infty$. For F, G, $\in L^\infty(\mathfrak{A})$, we write $F=G$ and $F\leqq G$ to mean $F(\omega)=G(\omega)$ a.e. and $F(\omega)\leqq G(\omega)$ a.e.

The algebra $L^\infty(\mathfrak{A})$ is isomorphic and isometric with the algebra $R(X)$ of all continuous functions on a compact Hausdorff space X, either via the Gelfand theorem or via a direct proof [29]. The space X may be characterized as the Boolean space of the measure algebra $\mathfrak{A}$. Since $\mathfrak{A}$ is a complete Boolean algebra [19, Lemma 3.2.1], the space X is extremally disconnected. The isomorphism of $L^\infty(\mathfrak{A})$ with $R(X)$ enables us to define the concept of a relatively complete subalgebra of $L^\infty(\mathfrak{A})$; we shall not pause to do this explicitly.

8.1. Theorem. *Let S be a uniformly closed subalgebra of $L^\infty(\mathfrak{A})$ containing constant functions. The following are equivalent conditions on S:* (a) *S is a relatively complete subalgebra of $L^\infty(\mathfrak{A})$.* (b) *$S=L^\infty(\mathfrak{B})$, where $\mathfrak{B}$ is a Borel subfield of $\mathfrak{A}$.* (c) *S is closed under the pointwise convergence of sequences in S.*

Proof. A family S of functions on Ω is closed under pointwise convergence of sequences if $F_n\in S$ and $F_n(\omega)\rightarrow F(\omega)$ for all $\omega\in\Omega$ imply $F\in S$. That (b) and (c) are equivalent is a classical result [11, Chapter 9]. We will show that (a) implies (c) and that (b) implies (a).

To see that (b) implies (a), we first remark that if $\mathfrak{B}$ is a Borel subfield of $\mathfrak{A}$, then $(\Omega, \mathfrak{B}, P)$ is a probability space, and hence its measure algebra $\mathfrak{B}=\mathfrak{B}/\mathfrak{B}\cap\mathfrak{I}$ is complete. Then its representation space Y is extremally disconnected, and hence $R(Y)$ is a conditionally complete lattice. Therefore $L^\infty(\mathfrak{B})$ is a relatively complete subalgebra. All of this follows from Theorem 5.3.

To prove that (a) implies (c), it is well-known that it is sufficient to prove that S is closed under monotone decreasing limits of sequences, since S is a uniformly closed algebra containing constants. Then, let $F_n\in S$, and suppose $F_n(\omega)\downarrow F(\omega)$ for all $\omega\in\Omega$. Let $G\in S$ be the least element in S satisfying $G\geqq F$. Then $F_n\geqq G$, and hence $F\geqq G$. Thus $F=G$, and hence $F\in S$.

The appearance of a Borel subfield $\mathfrak{B}$ of $\mathfrak{A}$ leads us at once to a consideration of the conditional expectation of a function, given the subfield $\mathfrak{B}$. We recall briefly the meaning of this concept (for details, see [10]). For any summable function F on the probability space $(\Omega, \mathfrak{A}, P)$ and for any $B\in\mathfrak{B}$, define $Q(B)=\int_B F(\omega)dP(\omega)$. This defines a finite signed measure on $\mathfrak{B}$, and hence, by the Radon-Nikodym Theorem, there is a summable function F^* on the probability space $(\Omega, \mathfrak{B}, P)$ such that $Q(B)=\int_B F^*(\omega)dP(\omega)$. The function F^* is called the conditional expectation of F given $\mathfrak{B}$, and is denoted by $E(F|\mathfrak{B})$. We note that if $F\in L^\infty(\mathfrak{A})$, then $E(F|\mathfrak{B})\in L^\infty(\mathfrak{B})$. First observe that if F is measurable $(\mathfrak{B})$, then $E(F|\mathfrak{B})=F$. Hence if $-\alpha\leqq F\leqq\alpha$, then $-\alpha\leqq E(F|\mathfrak{B})\leqq\alpha$, so that $F\in L^\infty(\mathfrak{A})$ implies $E(F|\mathfrak{B})\in L^\infty(\mathfrak{B})$.

8.2. Definition. Let $\mathfrak{B}$ be a subfield of $\mathfrak{A}$ in the probability space $(\Omega, \mathfrak{A}, P)$, and let $F\in L^\infty(\mathfrak{A})$. By the conditional maximum of F given $\mathfrak{B}$, we mean the least function $G\in L^\infty(\mathfrak{B})$ satisfying $F\leqq G$, and by the conditional minimum of F given $\mathcal{B}$ we mean the greatest function $H\in L^\infty(\mathcal{B})$ satisfying $H\leqq F$. We write $G=\max(F|\mathcal{B})$ and $H=\min(F|\mathcal{B})$. A conditional mean for $L^\infty(\mathcal{A})$ given $\mathcal{B}$ is a linear transformation μ from $L^\infty(\mathcal{A})$ into $L^\infty(\mathcal{B})$ such that $\min(F|\mathcal{B})\leqq\mu F\leqq\max(F|\mathcal{B})$ for each $F\in L^\infty(\mathcal{A})$. A conditional homomorphic mean is a mean which is also a homomorphism.

8.3. THEOREM. *Let μ be a linear transformation of $L^\infty(\mathcal{A})$ onto $L^\infty(\mathcal{B})$, where $\mathcal{B}$ is a Borel subfield of $\mathcal{A}$. Then μ is a conditional mean given $\mathcal{B}$ if and only if μ is order-preserving and $\mu E(F|\mathcal{B})=E(F|\mathcal{B})$ for all $F\in L^\infty(\mathcal{A})$.*

This follows at once from Theorem 6.3. We observe that since the conditional expectation itself has these properties, it is a conditional mean. It is clearly not a homomorphic mean in general. We proceed now to show that $L^\infty(\mathcal{A})$ is "rich" in conditional homomorphic means.

8.4. THEOREM. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $\mathcal{B}$ be any subfield of $\mathcal{A}$. For any given $F\in L^\infty(\mathcal{A})$, there is a conditional homomorphic mean μ for $L^\infty(\mathcal{A})$ given $\mathcal{B}$ such that $\mu F=\max(F|\mathcal{B})$.*

Proof. Represent $L^\infty(\mathcal{A})$ as $R(X)$, where X is the Boolean space of the measure algebra $\mathfrak{A}$. Let B be the closed subalgebra of $R(X)$ which is the image of $L^\infty(\mathcal{B})$ in the representation, and let ϕ, π, Y be as in §3. We know from §5 that X and Y are extremally disconnected, and that ϕ is a Boolean relation. Then ϕ determines a quantifier $\exists$ in the complete Boolean algebra $E(X)$ of idempotents in $R(X)$. The range of $\exists$ is a Boolean algebra isomorphic with $E(Y)$, which is also complete. By a theorem of Gleason [6], Sikorski [20], and Halmos [9], we know that for any $e\in E(X)$, there exists a cross-section γ: $Y\to X$ of the projection π: $X\to Y$ such that $\exists e(x)=e(\gamma\pi x)$, for all $x\in X$. Since $\exists e=Me$ for any $e\in E(X)$, the conclusion of the theorem is true for any $e\in E(X)$. Moreover, since $me=1-M(1-e)$ for each $e\in E(X)$, it follows that there is also a cross-section γ such that $me(x)=e(\gamma\pi x)$ for all x. If $\alpha\geqq 0$ then $M(\alpha e)=\alpha Me$, while if $\alpha<0$ then $M(\alpha e)=-m(-\alpha e)$. It follows that the conclusion holds for any scalar multiple of any element in $E(X)$.

Suppose f_1 and f_2 are two functions in $R(X)$ for which the conclusion holds: there exist cross-sections γ_i such that $f_i(\gamma_i\pi x)=Mf_i(x)$, $i=1, 2$. Let $U=\{x\in X: Mf_1(x)<Mf_2(x)\}$; then U is open, so that $\overline{U}$ is an open-closed set. Since $\pi^{-1}\pi U=U$, then $\pi^{-1}\pi\overline{U}=\overline{U}$, because π is an open mapping. Let $V=\pi\overline{U}$; then V is open-closed in Y. Define γ: $Y\to X$ by setting $\gamma(y)=\gamma_2(y)$ if $y\in V$, and by setting $\gamma(y)=\gamma_1(y)$ if $y\in Y-V$. Since V and $Y-V$ are both open-closed, γ is continuous. It is clearly a cross-section. Moreover, $M(f_1\vee f_2)(x)=Mf_1(x)\vee Mf_2(x)$ for all $x\in X$. If $x\in\overline{U}$, then $Mf_1(x)\leqq Mf_2(x)$, so $M(f_1\vee f_2)(x)=Mf_2(x)$. But if $x\in\overline{U}$, then $\pi x\in V$, and hence $\gamma\pi x=\gamma_2\pi x$. Then $(f_1\vee f_2)(\gamma\pi x)=(f_1\vee f_2)(\gamma_2\pi x)=f_1(\gamma_2\pi x)\vee f_2(\gamma_2\pi x)=f_1(\gamma_2\pi x)\vee Mf_2(x)\leqq Mf_1(x)\vee Mf_2(x)=Mf_2(x)$. Hence the one inequality in this chain collapses to equality, and we conclude that $(f_1\vee f_2)(\gamma\pi x)=M(f_1\vee f_2)(x)$ if $x\in\overline{U}$. A similar argument shows that the same equality holds for all $x\in X-\overline{U}$. We have thus

proved that the set of all functions for which the conclusion of the theorem holds is a set closed under formation of the least upper bound of a pair in the set, and that the set contains all scalar multiples of elements in $E(X)$. It is immediate and well-known that this is therefore a set containing the algebra generated in $R(X)$ by $E(X)$. In other words, any "simple" function $f=\sum_{j=1}^{n}\alpha_j e_j$, where α_j is real and $e_j\in E(X)$, has a homomorphic mean μ such that $\mu f=Mf$.

Given any function $f\in R(X)$ and any $n>0$, there is a simple function f_n in the above sense such that $\|f_n-f\|<1/2n$. Let μ_n be a homomorphic mean such that $\mu_n f_n = Mf_n$. Then $\|Mf-\mu_n f\| \leqq \|Mf-Mf_n\| + \|Mf_n-\mu_n f_n\| + \|\mu_n f_n-\mu_n f\|$. Then $\|Mf-Mf_n\|<1/2n$, by 4.4, and $\|Mf_n-\mu_n f_n\|=0$. By 6.2, $\|\mu_n\|=1$, so that $\|\mu_n f_n-\mu_n f\|<1/2n$. Hence $\|Mf-\mu_n f\|<1/n$. In terms of the original algebra $L^\infty(\mathfrak{A})$, we have shown that for any $F\in L^\infty(\mathfrak{A})$, there exists a sequence μ_n of conditional homomorphic means such that $\|\max(F|\mathfrak{B})-\mu_n F\|_\infty <1/n$.

Consider the family of all complex-valued essentially bounded measurable functions on $(\Omega, \mathfrak{B}, P)$, and represent it as a family of bounded operators on the complex Hilbert space of square-summable functions on $(\Omega, \mathfrak{B}, P)$, in the usual way [19, p. 290]. This family is then a maximal abelian ring of operators on the Hilbert space [19, Theorem 5.1]. Then the unit sphere is compact in the weak operator topology [16]. Since the taking of adjoints is continuous in the weak operator topology, it follows that the set $U^\infty(\mathfrak{B}) = \{F\in L^\infty(\mathfrak{B}): \|F\|_\infty\leqq 1$ is compact in this topology. Let $\mathfrak{K}$ be the set of all functions from $U^\infty(\mathfrak{A})$ into $U^\infty(\mathfrak{B})$, in the product topology. This is a compact Hausdorff space. For any two reals α, β and any two F, $G\in U^\infty(\mathfrak{A})$, the sets $\{\Phi\in\mathfrak{K}: \Phi(FG)=\Phi(F)\cdot\Phi(G)\}$ and $\{\Phi\in\mathfrak{K}: \Phi(\alpha F+\beta G)=\alpha\Phi(F)+\beta\Phi(G)\}$ are closed and hence compact. It follows, by intersecting, that the set of all homomorphisms of $L^\infty(\mathfrak{A})$ into $L^\infty(\mathfrak{B})$ is compact.

Consider the sequence μ_n of homomorphic means obtained above. Let $\mathfrak{H}_n$ denote the closure in $\mathfrak{K}$ of the subsequence $\{\mu_n, \mu_{n+1}, \cdots\}$. The set $\mathfrak{H}_n$ consists of homomorphisms of $L^\infty(\mathfrak{A})$ into $L^\infty(\mathfrak{B})$. Moreover, if $\mathfrak{H}=\bigcap_{n=1}^{\infty}\mathfrak{H}_n$, then $\mathfrak{H}$ is not empty. Let $\mu\in\mathfrak{H}$; we will show that $\mu F=\max(F|\mathfrak{B})$. Given $\epsilon>0$, choose n such that $2<n\epsilon$. Since $\mu\in\mathfrak{H}_n$, then for any given weak operator neighborhood of μF, there is some $k\geqq n$ such that $\mu_k F$ belongs to this neighborhood. Let H_1, H_2 be square-summable, complex-valued functions on $(\Omega, \mathfrak{B}, P)$, with $\|H_i\|_2=1$. Then there is a $k\geqq n$ such that $|((\mu_k F-\mu F)H_1, H_2)|<1/n$. Then we have $|((\max(F|\mathfrak{B})-\mu F)H_1, H_2)| \leqq |((\max(F|\mathfrak{B})-\mu_k F)H_1, H_2)| + |((\mu_k F-\mu F)H_1, H_2)| < \|\max(F|\mathfrak{B})-\mu_k F\|_\infty \|H_1\|_2\cdot\|H_2\|_2+1/n<1/k+1/n \leqq 2/n<\epsilon$. This shows that $\max(F|\mathfrak{B})-\mu F=0$, as desired. It remains to show that μ is actually a mean. We do this by establishing the condition of 8.3. Let $G\in L^\infty(\mathfrak{A})$ be arbitrary, H_1, H_2 as before, and $\epsilon>0$. Then there is an integer $k\geqq n$ such that $|((\mu_k E(G|\mathfrak{B})-\mu E(G|\mathfrak{B}))H_1, H_2)|<\epsilon$. Since $\mu_k E(G|\mathfrak{B}) =E(G|\mathfrak{B})$, we have $E(G|\mathfrak{B})-\mu E(G|\mathfrak{B})=0$. Hence the proof is complete.

9. **Invariant means.** Let T be a measurable transformation on a probabil-

ity space $(\Omega, \mathfrak{A}, P)$. We call T absolutely continuous if $P(E)=0$ implies $P(T^{-1}E)=0$, and we call T incompressible if $E\subset T^{-1}E$ implies $P(T^{-1}E-E)=0$. Throughout this section, all transformations will be assumed to be absolutely continuous.

Call a set $B\in\mathfrak{A}$ invariant if and only if $P(B+T^{-1}B)=0$, where $+$ denotes symmetric difference. The set $\mathfrak{B}$ of all invariant sets is a Boolean subfield of $\mathfrak{A}$, containing all null sets. For any function F on Ω which is measurable $(\mathfrak{A})$, the function UF defined by $UF(\omega)=F(T\omega)$ is also measurable $(\mathfrak{A})$. The absolute continuity of T implies that U maps $L^\infty(\mathfrak{A})$ into itself. We shall henceforth let U denote the restriction to $L^\infty(\mathfrak{A})$. A function $F\in L^\infty(\mathfrak{A})$ is called invariant if and only if $F\in L^\infty(\mathfrak{B})$.

The previous section applies without change to $L^\infty(\mathfrak{A})$ and $L^\infty(\mathfrak{B})$, and we shall preserve the notation of that section here. It is only necessary to adjoin the corresponding objects for T to that account. The mapping T^{-1} defined on $\mathfrak{A}$ by T is a σ-endomorphism of $\mathfrak{A}$. Since T is absolutely continuous, T^{-1} induces a σ-endomorphism of the measure algebra $\mathfrak{A}=E(X)$. This in turn induces a continuous transformation t of X into itself, and t induces a linear transformation u of $R(X)$ into itself. For any $f\in R(X)$, $uf(x)=f(tx)$, and u restricted to $E(X)$ is the σ-endomorphism induced by T^{-1}.

Let B be the subalgebra of $R(X)$ corresponding to $L^\infty(\mathfrak{B})$. Then the set of idempotents in B defines a quantifier $\exists$ in the Boolean algebra $E(X)$. It is shown in [26, §7] that $\exists e=\bigvee_{n=1}^\infty u^n e$ if and only if T is incompressible. This fact is the generalized Poincaré Recurrence Theorem [26, Theorem 14]. (See [28] for this result without absolute continuity.) In another form, this says that T is incompressible if and only if $\bigvee_{n=1}^\infty u^n e=\bigvee_{n=m}^\infty u^n e$ for each $e\in E(X)$ and each integer $m\geqq 1$ [26, Theorem 13]. This last condition is equivalent to stating that $\bigcup_{n=1}^\infty T^{-n}A$ is an invariant set for each $A\in\mathfrak{A}$.

9.1. THEOREM. *If T is absolutely continuous, then T is incompressible if and only if* $\max(F|\mathfrak{B})=\mathrm{lub}_{1\leqq n<\infty} U^nF$ *for every* $F\in L^\infty(\mathfrak{A})$.

Proof. Set $H(\omega)=\mathrm{lub}_{0\leqq n<\infty} F(T^n\omega)$ and set $G=\max(F|\mathfrak{B})$. Clearly $H\leqq G$. Let $A=\{\omega\in\Omega: H(\omega)>\lambda\}$; then $A=\bigcup_{n=1}^\infty\{\omega\in\Omega: F(T^n\omega)>\lambda\}=\bigcup_{n=1}^\infty T^{-n}A$. Then, if T is incompressible A is invariant. Therefore $H\in L^\infty(\mathfrak{B})$, so that $G\leqq H$. Therefore $G=H$, and $H(\omega)=H(T\omega)$ a.e. Since $H(T\omega)=\mathrm{lub}_{1\leqq n<\infty}F(T^n\omega)$, half of the theorem is proved. The other half is immediate from the above remarks.

9.2. DEFINITION. An invariant mean in $L^\infty(\mathfrak{A})$ for an absolutely continuous transformation T is a conditional mean given the subfield $\mathfrak{B}$ of invariant sets, such that $\mu UF=\mu F$ for each $F\in L^\infty(\mathfrak{A})$.

9.3. THEOREM. *Let T be an absolutely continuous transformation on a probability space* $(\Omega, \mathfrak{A}, P)$. *Then there exists an invariant mean μ in $L^\infty(\mathfrak{A})$ for T.*

Proof. Let μ be any conditional mean given $\mathfrak{B}$; by Theorem 8.4 such conditional means exist. If $E(F|\mathfrak{B})$ is the conditional expectation of F, then

$UE(F|\mathcal{B}) = E(F|\mathcal{B})$, and hence $\mu UE(F|\mathcal{B}) = \mu E(F|\mathcal{B}) = E(F|\mathcal{B})$, by 8.3. Moreover, it is clear that $F \leqq G$ implies $UF \leqq UG$, and hence that $\mu UF \leqq \mu UG$. Therefore μU is again a mean. We have remarked earlier that the collection of all means forms a convex set in the algebra of all linear transformations of $L^\infty(\mathcal{A})$ into itself, and that this set is compact in the weak operator topology. It is immediate that the mapping which sends μ onto μU is a continuous mapping in the weak operator topology. By the Tychonoff Fixed-Point Theorem [24] there is a mean μ_0 such that $\mu_0 U = \mu_0$. Hence μ_0 is an invariant mean.

9.4. THEOREM. *If T is an absolutely continuous transformation on a probability space $(\Omega, \mathcal{A}, P)$, then there exists a finitely additive measure Q on $\mathcal{A}$ with the following properties*: (a) $P(E) = 0$ *implies* $Q(E) = 0$; (b) $P(E) = Q(E)$ *if E is invariant*; (c) $Q(T^{-1}E) = Q(E)$ *for all* $E \in \mathcal{A}$.

Proof. Let μ be an invariant mean in $L^\infty(\mathcal{A})$ for T. For each $E \in \mathcal{A}$, define $Q(E) = \int_\Omega \mu\chi_E(\omega)dP(\omega)$; this is the desired measure.

It has been shown that if T is absolutely continuous and if the conclusion of Birkhoff's Individual Ergodic Theorem holds for characteristic functions of measurable sets, then there exists a *countably* additive measure Q with the properties of this theorem [27]. Furthermore, if T is incompressible, then this countably additive measure Q has the further property that $Q(E) = 0$ implies $P(E) = 0$. This raises the question of whether incompressibility can be related to this condition for finitely additive measures Q as in 9.4. It is trivial that if any such Q has this property, that $Q(E) = 0$ implies $P(E) = 0$, then T is incompressible. An even weaker assumption which guarantees the incompressibility of T is that for any set $E \in \mathcal{A}$ with $P(E) \neq 0$ there is a finitely additive invariant measure Q such that $Q(E) \neq 0$. Is the converse true? This can be phrased more felicitously: define the radical of an absolutely continuous T to be the set of all $E \in \mathcal{A}$ such that for any finitely additive measure Q satisfying the conditions of 9.4 we have $Q(E) = 0$. If the radical of T is precisely the σ-ideal of null sets, then T is incompressible. If T is incompressible, does its radical consist only of null sets?

The classical invariant measure problem is to give necessary and sufficient conditions for the existence of finite or σ-finite, equivalent, invariant measures. (See [8] for a discussion of the problem.) There exist incompressible transformations which admit no finite [8, p. 85] and no σ-finite [17] measures of this kind. Theorem 9.4 furnishes an affirmative counterpart to these negative answers.

As a final comment, we may remark that the invariant Q of 9.4 is never a trivial measure, because of condition (b). We may also note that, even if T is incompressible, the Hurewicz Ergodic Theorem [12] cannot be used to produce the finitely additive Q in the same manner in which Birkhoff's theorem is used in [27] for a countably additive measure.

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Tulane University of Louisiana,
New Orleans, Louisiana